Final exam Robust Control

30 October 2025, 11:45–13:45

The exam consists of 3 exercises. Please write clearly and provide motivations for all your answers. You get 4 points for free and the maximum possible number of points is 40. Your grade is equal to the number of points divided by 4. Good luck!

1 $(3 + 2 + 2 + 5 = 12$ points) **H_2 optimal control**

Consider the system Σ of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \;\; z(t) = Cx(t) + Du(t).$$

Here, $u(t)$ is the control input, $d(t)$ the disturbance input, and $z(t)$ the performance output.

(a) Give the precise definition of the H_2 optimal control problem by state feedback.

(b) Assume that (A, B) is stabilizable and the problem is in standard form. Explain how to compute the optimal performance.

(c) Under which additional condition does there exist an optimal state feedback? Explain how to compute one in this case.

(d) Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (u + d), \quad z = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

For all values of $\alpha, \beta \in \mathbb{R}$, compute (if they exist) the optimal performance and an optimal controller.

2 $(2 + 3 + 4 + 3 = 12$ points) **Linear matrix inequalities**

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $n > m$ and $\operatorname{rank} B = m$. The goal of this exercise will be to show that (A, B) is stabilizable if and only if the linear matrix inequality

$$AP + PA^\top - BB^\top < 0 \tag{1}$$

has a symmetric positive definite solution $P \in \mathbb{R}^{n \times n}$.

(a) Give a precise statement of the Hautus test for stabilizability.

(b) Assume that $P > 0$ satisfies (1). Show that (A, B) is stabilizable.

(c) Let $Q \in \mathbb{R}^{n \times n}$ be symmetric. Show that the linear matrix inequality

$$Q + BX + (BX)^\top < 0$$

has a solution $X \in \mathbb{R}^{m \times n}$ if and only if there exists $\mu \in \mathbb{R}$ such that $Q - \mu BB^\top < 0$. *Hint*: apply Finsler's lemma.

(d) Finally, assume that (A, B) is stabilizable. As we have seen in the lectures, this means that there exists a stabilizing feedback gain $F \in \mathbb{R}^{m \times n}$ and a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ such that $(A + BF)P + P(A + BF)^\top < 0$. Prove that this implies that (1) has a solution $P > 0$.

3 $(5 + 3 + 4 = 12$ points) **H_∞ norm, small gain theorem**

For $i = 1, 2, \ldots, N$, consider the system Σ_i given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \;\; y_i(t) = C_i x_i(t) + D_i u_i(t).$$

Assume that A_i is Hurwitz and define the transfer matrix

$$G_i(s) = C_i(sI - A_i)^{-1} B_i + D_i.$$

Suppose that $\gamma_i > 0$ is such that $||G_i||_\infty < \gamma_i$.

(a) Let $N = 2$. Consider the series interconnection of Σ_1 and Σ_2 with input u_1 and output y_2, obtained by setting $u_2 = y_1$. Compute the transfer matrix of this interconnected system and provide a bound on its H_∞ norm.

(b) Let $N = 2$. Consider the feedback interconnection of Σ_1 and Σ_2 obtained by setting $u_1 = y_2$ and $u_2 = y_1$. Give a precise formulation of the small gain theorem for this interconnected system.

(c) Let $N \geq 2$. Consider the interconnected system $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_N$ obtained by setting $u_1 = y_N$, $u_{i+1} = y_i$ for $i = 1, 2, \ldots, N - 1$. Formulate and prove a theorem that provides conditions under which $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_N$ is asymptotically stable.

End of exam, cheers!